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LETTER TO THE EDITOR

On the Lie extended method in quantum physics and its supersymmetric version

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Abstract. We propose the supersymmetric version of the Lie extended method in quantum physics. We point out the main ideas through the explicit example of the one-dimensional supersymmetric harmonic oscillator.

We are concerned in this letter with the description of *supersymmetric* quantum systems, i.e. those which have both bosonic and fermionic degrees of freedom (Witten 1981). More precisely, let us consider a wave equation of the form

$$\Delta\Phi \equiv (\mathrm{i}\partial_t - H^{\mathrm{ss}})\Phi = 0 \tag{1}$$

where H^{ss} is the supersymmetric Hamiltonian obtained from the conserved Q-type supercharges generating the superalgebra

$$H^{\mathrm{ss}} = \{Q, Q^\dagger\} \tag{2}$$

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \qquad [H^{\mathrm{ss}}, Q] = [H^{\mathrm{ss}}, Q^\dagger] = 0. \tag{3}$$

If we limit ourselves for simplicity to the so-called $N = 2$ supersymmetric quantum mechanical context, we can illustrate our considerations through the one-dimensional harmonic oscillator (Witten 1981, Salomonson and Van Holten 1982, De Crombrugghe and Rittenberg 1983), a well known application (Beckers and Hussin 1986, Beckers *et al* 1987, 1988) when (super)symmetries are concerned. In this case, the operator Δ in (1) takes the explicit form

$$\Delta \equiv \Delta(t, x, p, \Psi, \bar{\Psi}) \tag{4}$$

$$\equiv \mathrm{i}\partial_t + \tfrac{1}{2}\partial_x^2 - \tfrac{1}{2}\omega^2 x^2 - \frac{\omega}{2}[\Psi, \bar{\Psi}] \tag{5}$$

where as usual we maintain the angular frequency ω while the mass is taken as unity. We recall in (4) the dependence of Δ in terms of the (commuting) *bosonic* operators x and $p = -\mathrm{i}\partial_x$ as well as in terms of the (anticommuting) *fermionic* operators Ψ and $\bar{\Psi}$ such that

$$[p, x] = -\mathrm{i} \qquad \{\Psi, \bar{\Psi}\} = \mathrm{I} \qquad \{\Psi, \Psi\} = \{\bar{\Psi}, \bar{\Psi}\} = 0. \tag{6}$$

Now let us study the invariance condition

$$[\Delta, X] = \lambda\Delta. \tag{7}$$

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It corresponds to the invariance of (1) with $\Delta \equiv (5)$ under the infinitesimal transformation $1+\mathrm{i}\varepsilon X$ where we require that ε be a supernumber (Berezin 1966) and X is the linear differential operator given by

$$X = X_{\bar{0}} + X_{\bar{1}}. \tag{8}$$

By having (8) in this form, we explicitly refer to a graduation of the invariance algebra we are searching for, $X_{\bar{0}}$ and $X_{\bar{1}}$ being respectively defined as the even and odd parts of the generator X. The formulation of such a problem will correspond (at the level of the algebra) to the supersymmetric version of the Lie extended method (Lie 1891, 1922–1927, Ovsiannikov 1982, Olver 1986) for the determination of the maximal Lie symmetry group, leaving invariant a well defined (classical or) quantum differential equation (Niederer 1972, 1973, Rudra 1984, 1986, Fushchich and Nikitin 1987).

Up to a supplementary comment on the function λ (which will be presented at the end of this letter), we immediately get from (7) and (8) the set of conditions

$$[\Delta, X_{\bar{0}}] = \lambda_{\bar{0}}(t,x)\Delta \tag{9a}$$

and

$$[\Delta, X_{\bar{1}}] = 0 \tag{9b}$$

where $\lambda_{\bar{0}}$ is an arbitrary even function. Equation (9a) expresses the usual context previously considered by other authors (see for example Niederer 1973) for the quantum harmonic oscillator *but* it is generalised by including, in $X_{\bar{0}}$, even operators associated with the (even) products $\Psi\bar{\Psi}$, $\partial_\Psi\partial_{\bar{\Psi}}$, $\Psi\partial_\Psi$, $\bar{\Psi}\partial_{\bar{\Psi}}$, $\Psi\partial_{\bar{\Psi}}$ and $\bar{\Psi}\partial_\Psi$. Equation (9b) is new and refers, in $X_{\bar{1}}$, to the single operators Ψ, $\bar{\Psi}$, ∂_Ψ and $\partial_{\bar{\Psi}}$ which evidently show an odd character. The new odd operators ∂_Ψ and $\partial_{\bar{\Psi}}$ are introduced for convenience and parallelism in notation with respect to the bosonic context: we just require that they be respective anticommuting operators satisfying in correspondence with (6):

$$\{\Psi, \partial_\Psi\} = \mathbf{I} = \{\bar{\Psi}, \partial_{\bar{\Psi}}\}$$
and
$$\{\Psi, \partial_{\bar{\Psi}}\} = 0 = \{\bar{\Psi}, \partial_\Psi\}. \tag{10}$$

We are thus concerned with the following even and odd parts of our general X-generator:

$$X_{\bar{0}} = a(t,x) + a_0(t,x)\partial_t + a_1(t,x)\partial_x + b(t,x)\Psi\bar{\Psi} + c_1(t,x)\Psi\partial_\Psi + c_2(t,x)\bar{\Psi}\partial_{\bar{\Psi}} + d_1(t,x)\bar{\Psi}\partial_\Psi + d_2(t,x)\Psi\partial_{\bar{\Psi}} + e(t,x)\partial_\Psi\partial_{\bar{\Psi}} \tag{11a}$$

and

$$X_{\bar{1}} = \alpha_1(t,x)\Psi + \alpha_2(t,x)\bar{\Psi} + \beta_1(t,x)\Psi\partial_x + \beta_2(t,x)\bar{\Psi}\partial_x + \gamma_1(t,x)\partial_\Psi + \gamma_2(t,x)\partial_{\bar{\Psi}} + \delta_1(t,x)\partial_\Psi\partial_x + \delta_2(t,x)\partial_{\bar{\Psi}}\partial_x. \tag{11b}$$

By solving the system (9a) with the operators Δ and $X_{\bar{0}}$ given respectively by (5) and (11a), we are led to twelve *even* generators of one-parameter substructures. Seven of them can be called, for future reference and direct connection with already known results (Beckers and Hussin 1986):

$$H_B \equiv \tfrac{1}{2}(p^2+\omega^2x^2) \qquad C_\pm \equiv \mathrm{e}^{\mp 2\mathrm{i}\omega t}(p\pm \mathrm{i}\omega x)^2$$
$$H_F \equiv \tfrac{1}{2}[\Psi, \bar{\Psi}] \qquad P_\pm \equiv \mathrm{e}^{\mp \mathrm{i}\omega t}(p\pm \mathrm{i}\omega x) \qquad \mathbf{I}. \tag{12a}$$

The five supplementary generators take the following forms:

$$\begin{cases} X_1 \equiv e^{i\omega t}\bar{\Psi}\partial_\Psi \\ X_2 \equiv e^{-i\omega t}\Psi\partial_{\bar{\Psi}} \end{cases} \qquad X_3 \equiv \Psi\partial_\Psi - \bar{\Psi}\partial_{\bar{\Psi}} + \partial_\Psi\partial_{\bar{\Psi}} \qquad \begin{cases} X_4 \equiv e^{-i\omega t}(\Psi\partial_\Psi - \Psi\bar{\Psi}) \\ X_5 \equiv e^{i\omega t}(\bar{\Psi}\partial_{\bar{\Psi}} - \bar{\Psi}\Psi) \end{cases}. \tag{12b}$$

Moreover the system ($9b$) with Δ and $X_{\bar{\imath}}$ given respectively by (5) and ($11b$) leads to twelve *odd* generators. Here they can be arranged in the following six *pairs*

$$\begin{cases} Q_+ \equiv (p - i\omega x)\Psi \\ Q_- \equiv (p + i\omega x)\bar{\Psi} \end{cases} \qquad \begin{cases} S_+ \equiv e^{-2i\omega t}(p + i\omega x)\Psi \\ S_- \equiv e^{2i\omega t}(p - i\omega x)\bar{\Psi} \end{cases} \qquad \begin{cases} T_+ \equiv e^{-i\omega t}\Psi \\ T_- \equiv e^{i\omega t}\bar{\Psi} \end{cases} \tag{12c}$$

and

$$\begin{cases} X_6 \equiv \partial_\Psi - \bar{\Psi} \\ X_7 \equiv \partial_{\bar{\Psi}} - \Psi \end{cases} \qquad \begin{cases} X_8 \equiv (1/\sqrt{2})\, e^{i\omega t}(p - i\omega x)(\partial_\Psi - \bar{\Psi}) \\ X_9 \equiv (1/\sqrt{2})\, e^{-i\omega t}(p + i\omega x)(\partial_{\bar{\Psi}} - \Psi) \end{cases}$$
$$\begin{cases} X_{10} \equiv (1/\sqrt{2})\, e^{-i\omega t}(p + i\omega x)(\partial_\Psi - \bar{\Psi}) \\ X_{11} \equiv (1/\sqrt{2})\, e^{i\omega t}(p - i\omega x)(\partial_{\bar{\Psi}} - \Psi). \end{cases} \tag{12d}$$

At this stage, we can recover a recent result (Beckers and Hussin 1986) corresponding to the *largest* (kinematical *or* dynamical) invariance superalgebra of the one-dimensional harmonic oscillator, a study which has also been extended to the general context of n-dimensional harmonic oscillators and their dynamical as well as kinematical invariance superalgebras (Beckers *et al* 1987, 1988). Indeed we get the superalgebra osp(2/2)□ sinh(2/2) generated by the operators ($12a$) *and* ($12c$) *if* we trivialise the operators ($12b$) and ($12d$). This remarkable fact is immediately realised if we characterise our fermionic variables as the Pauli 2×2 matrices according to

$$\Psi \equiv \sigma_+ \qquad \bar{\Psi} \equiv \sigma_- \tag{13a}$$

while the anticommuting operators ∂_Ψ and $\partial_{\bar{\Psi}}$ have to be identified such as

$$\partial_\Psi \equiv \sigma_- \qquad \partial_{\bar{\Psi}} \equiv \sigma_+. \tag{13b}$$

Furthermore we deduce the additional information (with respect to (6) and (10)):

$$\{\partial_\Psi, \partial_{\bar{\Psi}}\} = \mathbb{I} \qquad \{\partial_\Psi, \partial_\Psi\} = 0 = \{\partial_{\bar{\Psi}}, \partial_{\bar{\Psi}}\}. \tag{14}$$

We effectively notice that all the operators $X_1, \ldots, X_{11}$ given by ($12b$) and ($12d$) are identically zero with the choices (13). We also stress the point that the Q-type (Witten 1981) *and* S-type (Fubini and Rabinovici 1984) supercharges given in ($12c$) have the expected expressions (Beckers and Hussin 1986).

The wealth of the above results can now be shown by noticing that the first constraint in (14) can be rewritten as

$$\{\partial_\Psi, \partial_{\bar{\Psi}}\} = \frac{d}{2}\mathbb{I} \tag{15}$$

with $d = 2$, the dimension of the Pauli matrices. We can thus propose higher dimensions ($d = 4, \ldots$) of matrix realisations so that the generators $X_1, \ldots, X_{11}$ could be non-trivial (by opposition to the case when $d = 2$).

Let us take here an explicit example. By choosing $d = 4$, we propose the fermionic associations:

$$\Psi \equiv \sigma_+ \otimes \mathbb{I}_2 \qquad \bar{\Psi} \equiv \sigma_- \otimes \mathbb{I}_2 \tag{16a}$$

and

$$\partial_\Psi \equiv \sigma_- \otimes \mathbb{I}_2 + \sigma_3 \otimes \sigma_- \qquad \partial_{\bar{\Psi}} \equiv \sigma_+ \otimes \mathbb{I}_2 + \sigma_3 \otimes \sigma_+ \tag{16b}$$

where I_2 is the 2×2 identity matrix. It is then straightforward to verify the relations (6), (10) and (15) with the choices (16) and to notice that the eleven generators (12*b*) and (12*d*) are now non-trivial. In this case we get *twenty four* generators. We can show that they close under commutation between even–even and even–odd generators and under anticommutation between odd–odd ones. The corresponding superstructure is now the non-simple superalgebra osp(4/2)□ sinh(4/2). We thus obtain a remarkable set of new conserved constants of motion associated with the invariance of the supersymmetric wave equation of the one-dimensional harmonic oscillator.

Many points can already be made from these results.

In general, the arbitrary function λ contained in (7) can be a superfunction. If we introduce

$$\lambda(t, x, \Psi, \bar{\Psi}) = \lambda_{\bar{0}}(t, x) + \lambda_{\bar{1}}(t, x)(\Psi + \bar{\Psi}) \tag{17}$$

we have in correspondence with (9):

$$[\Delta, X_{\bar{0}}] = \lambda_{\bar{0}}\Delta \qquad [\Delta, X_{\bar{1}}] = \lambda_{\bar{1}}(\Psi + \bar{\Psi})\Delta \tag{18}$$

so that the main modification appears in (9*b*). A detailed calculation shows that $\lambda_{\bar{1}}$ is equal to zero so that our previous formulation of the problem is sufficient.

We also want to mention that the explicit constructions of the even $X_{\bar{0}}$- and odd $X_{\bar{1}}$-parts of the generator X are dictated here by searching for a *closed* superstructure as invariance superalgebra: such a demand restricts the order of admissible derivatives by taking account of $\partial_t \sim \partial_x^2$ as expressed by (5). Moreover we notice that (15) permits us to get a larger (closed) superstructure by enlarging matrix (and evidently Clifford) dimensions (Sattinger and Weaver 1986). An enlightening discussion of such a point of view connected with closed *or* open structures such as the ones obtained by Fushchich and Nikitin (1987) will be developed elsewhere (Beckers and Debergh 1989).

Another point concerns the six *even* generators constructed from Grassmannian (odd) variables, i.e. the fermionic Hamiltonian H_F given in (12*a*) and the five operators $X_1 \ldots X_5$ given in (12*b*). Remarkably they generate a compact Lie subalgebra so(4) characterised by the following six generators:

$$L_1 \equiv X_5 - I - \frac{1}{\omega} H_F \qquad L_2 \equiv \tfrac{1}{2}(X_2 + X_4 - X_1 - X_3) \qquad L_3 \equiv \frac{i}{2}(X_1 + X_2 + X_3 + X_4)$$

and (19)

$$A_1 \equiv \frac{1}{\omega} H_F \qquad A_2 \equiv \tfrac{1}{2}(X_1 - X_2 - X_3 + X_4) \qquad A_3 \equiv \frac{i}{2}(X_3 + X_4 - X_1 - X_2)$$

leading to the *commutation* relations

$$[L_i, L_j] = i\varepsilon_{ijk} L_k \qquad [L_i, A_j] = i\varepsilon_{ijk} A_k \qquad [A_i, A_j] = i\varepsilon_{ijk} L_k. \tag{20}$$

Within the realisation (16), we get here the fundamental so(4)-representation. In particular we also notice that the above L-operators appear as the quantum counterpart of the 'super-rotations' recently discussed by Frydryszak (1989).

As a final comment, we can evidently extend the above considerations to n-dimensional supersymmetric harmonic oscillators and to other supersymmetric systems (D'Hoker *et al* 1988). All the known equations of supersymmetric *classical* mechanics or pseudomechanics (Berezin and Marinov 1975, 1977, Casalbuoni 1976) can also be reanalysed following our method.

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